

Running Coupling Constants of Fermions with Masses in Quantum Electro Dynamics and Quantum Chromo Dynamics

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Abstract

Based on a simple but effective regularization-renormalization method (RRM), the running coupling constants (RCC) of fermions with masses in quantum electrodynamics (QED) and quantum chromodynamics (QCD) are calculated by renormalization group equation (RGE). Starting at $Q = 0$ (Q being the momentum transfer), the RCC in QED increases with the increase of Q whereas the RCCs for different flavors of quarks with masses in QCD are different and they increase with the decrease of Q to reach a maximum at low Q for each flavor of quark and then decreases to zero at $Q \rightarrow 0$. The physical explanation is given.

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I. Introduction

In most literatures and textbooks, the running coupling constant (RCC) in quantum electrodynamics (QED) is usually given as (see, e.g. Ref. [1] and Eq. (48) below):

$$\alpha(Q) = \frac{\alpha}{1 - \frac{2\alpha}{3\pi} \ln \frac{Q}{m_e}} \quad (1)$$

where m_e is the electron mass and $\alpha = \frac{e^2}{4\pi}$. Eq. (1) is very important in physics for it unveils the monotonically enhancing behavior of electromagnetic coupling constant α in accompanying the increase of momentum transfer Q between two charged particles and shows the existence of Landau singularity at an extremely large Q . However, in our opinion, there are still three aspects that can be improved in this paper. (a) Besides electron, the contributions of other charged leptons and quarks can not be neglected. (b) While Eq. (1) is scale invariant, it ignores totally the particle mass effect which is also important at low Q region. (c) While the normalization in Eq. (1) is inevitably made at $\alpha(Q = m_e) = \alpha = (137.03599)^{-1}$, we prefer to renormalize it at the Thomson limit ($Q \rightarrow 0$) irrespective of the particle mass.

As for quantum chromodynamics (QCD), similarly, the RCC of quark is usually expressed for massless quark and so is independent of the flavor of quark. For instance, it reads [2] (We use the Bjorken-Drell metric throughout this paper.):

$$\alpha_s(Q) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda_{QCD}^2)} \quad (2)$$

where $\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f$ with $C_A = 3$ and n_f the number of flavors of quarks. The singularity of Q in Eq. (2), Λ_{QCD} , is an energy scale characterizing the confinement of quarks in QCD, $\Lambda_{QCD} \simeq 200 MeV$ experimentally.

While Eq. (2) successfully shows the asymptotic freedom of quarks at high Q , it is not so satisfying at low Q region, especially for heavy quarks. The mass of c or b quarks, let alone t quark, is much higher than Λ_{QCD} . In other words, the c (or b , or t) quark does not exist at low Q region beneath the threshold for creating $c\bar{c}$ (or $b\bar{b}$, or $t\bar{t}$) pair and the latter

is different for different flavor. Therefore, instead of Eq. (2), we need a new calculation of renormalization group equation (RGE) for RCC to discriminate different flavors of quarks. Evidently, it is necessary to take the mass of quark into account.

In recent years, based on the so-called derivative renormalization method in the literature [3-11], proposed by Ji-feng Yang [12], a simple but effective renormalization-regularization method (RRM) was used by Ni *et al* [13-17]. It is characterized as follows. When encountering a superficially divergent Feynman diagram integral (FDI) at one-loop level, we first differentiate it with respect to external momentum or mass parameter enough times until it becomes convergent. After performing integration with respect to internal momentum, we reintegrate it with respect to the parameter the same times to return to original FDI. Then instead of divergence, some arbitrary constants C_i ($i = 1, 2, \dots$) appear in FDI, showing the lack of knowledge about the model at quantum field theory (QFT) level under consideration. They can only be fixed by experiments or by some other deep reasons in theory. Since all constants are fixed at one-loop level, all previous steps can be repeated at next loop expansion. The new RRM has got rid of the explicit divergence, the counterterm, the bare parameter and the ambiguous (arbitrary) running mass scale μ quite naturally. In section II we will explain this method by calculating the RCC in QED [16,18] which also serves as the basis of the following sections. Then in Sec. III the relevant formulation of RGE for RCC in QCD is presented. The numerical results are given at Sec. IV. The final section V will contain a summary and discussion.

II. RGE calculation of RCC in QED

As is well known, there are three kinds of Feynman diagram integral (FDI) at one-loop level in QED.

1. Self-energy of electron with momentum p

The FDI for self-energy of electron reads ($e < 0$) [19-22]

$$\begin{aligned}
-i\Sigma(p) &= (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{ik^2} \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\nu \\
&= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{N}{D}
\end{aligned} \tag{3}$$

$$\begin{aligned}
\frac{1}{D} &= \frac{1}{k^2[(p-k)^2 - m^2]} = \int_0^1 \frac{dx}{[k^2 - 2p \cdot kx + (p^2 - m^2)x]^2} \\
N &= g_{\mu\nu} \gamma^\mu (\not{p} - \not{k} + m) \gamma^\nu = -2(\not{p} - \not{k}) + 4m.
\end{aligned}$$

We first perform a shift in momentum integration: $k \longrightarrow K = k - xp$, so that

$$-i\Sigma(p) = -e^2 \int_0^1 dx [-2(1-x) \not{p} + 4m] I \tag{4}$$

and concentrate on the logarithmically divergent integral

$$I = \int \frac{d^4K}{(2\pi)^4} \frac{1}{[K^2 - M^2]^2} \tag{5}$$

with

$$M^2 = p^2 x^2 + (m^2 - p^2)x.$$

A differentiation with respect to M^2 is enough to get

$$\frac{\partial I}{\partial M^2} = \frac{-i}{(4\pi)^2} \frac{1}{M^2}. \tag{6}$$

Thus

$$I = \frac{-i}{(4\pi)^2} [\ln M^2 + C_1] = \frac{-i}{(4\pi)^2} \ln \frac{M^2}{\mu_2^2} \tag{7}$$

carries an arbitrary constant $C_1 = -\ln \mu_2^2$. After integration with respect to the Feynman parameter x , one obtains

$$\begin{aligned}
\Sigma(p) &= A + B \not{p} \\
A &= \frac{\alpha}{\pi} m \left[2 - \ln \frac{m^2}{\mu_2^2} + \frac{(m^2 - p^2)}{p^2} \ln \frac{(m^2 - p^2)}{m^2} \right] \\
B &= \frac{\alpha}{4\pi} \left\{ \ln \frac{m^2}{\mu_2^2} - 3 - \frac{(m^2 - p^2)}{p^2} \left[1 + \frac{m^2 + p^2}{p^2} \ln \frac{(m^2 - p^2)}{m^2} \right] \right\}.
\end{aligned} \tag{8}$$

Using the chain approximation, one can derive the modification of electron propagator as

$$\frac{i}{\not{p} - m} \longrightarrow \frac{i}{\not{p} - m} \frac{1}{1 - \frac{\Sigma(p)}{\not{p} - m}} = \frac{iZ_2}{\not{p} - m_R} \quad (9)$$

$$Z_2 = (1 - B)^{-1} \simeq 1 + B \quad (10)$$

$$m_R = \frac{m + A}{1 - B} \simeq (m + A)(1 + B) \simeq m + \delta m$$

$$\delta m \simeq A + mB. \quad (11)$$

For a free electron, the mass shell condition $p^2 = m^2$ leads to

$$\delta m = \frac{\alpha m}{4\pi} (5 - 3 \ln \frac{m^2}{\mu_2^2}).$$

We want the parameter m in the Lagrangian still being explained as the observed mass, i.e., $m_R = m_{\text{obs}} = m$. So $\delta m = 0$ leads to $\ln \frac{m^2}{\mu_2^2} = \frac{5}{3}$, which in turn fixes the renormalization factor for wave function

$$Z_2 = 1 - \frac{\alpha}{3\pi}. \quad (12)$$

2. Photon self-energy — vacuum polarization

$$\Pi_{\mu\nu}(q) = -(-ie)^2 \text{Tr} \int \frac{d^4 k}{(2\pi)^4} \gamma_\mu \frac{i}{\not{k} - m} \gamma_\nu \frac{i}{\not{k} - \not{q} - m}. \quad (13)$$

Introducing the Feynman parameter x as before and performing a shift in momentum integration: $k \rightarrow K = k - xq$, we get

$$\Pi_{\mu\nu}(q) = -4e^2 \int_0^1 dx (I_1 + I_2) \quad (14)$$

where

$$I_1 = \int \frac{d^4 K}{(2\pi)^4} \frac{2K_\mu K_\nu - g_{\mu\nu} K^2}{(K^2 - M^2)^2} \quad (15)$$

with

$$M^2 = m^2 + q^2(x^2 - x) \quad (16)$$

is quadratically divergent while

$$I_2 = \int \frac{d^4 K}{(2\pi)^4} \frac{(x^2 - x)(2q_\mu q_\nu - g_{\mu\nu} q^2) + m^2 g_{\mu\nu}}{(K^2 - M^2)^2} \quad (17)$$

is only logarithmically divergent like that in Eqs. (5)–(7). An elegant way for handling I_1 is modifying M^2 into

$$M^2(\sigma) = m^2 + q^2(x^2 - x) + \sigma \quad (18)$$

and differentiating I_1 with respect to σ two times. After integration with respect to K , we reintegrate it with respect to σ two times, arriving at the limit $\sigma \rightarrow 0$:

$$I_1 = \frac{ig_{\mu\nu}}{(4\pi)^2} \{[m^2 + q^2(x^2 - x)] \ln \frac{m^2 + q^2(x^2 - x)}{\mu_3^2} + C_2\} \quad (19)$$

with two arbitrary constants: $C_1 = -\ln \mu_3^2$ and C_2 . Combining I_1 and I_2 together, we find

$$\Pi_{\mu\nu}(q) = \frac{8ie^2}{(4\pi)^2} (q_\mu q_\nu - g_{\mu\nu} q^2) \int_0^1 dx (x^2 - x) \ln \frac{m^2 + q^2(x^2 - x)}{\mu_3^2} - \frac{i4e^2}{(4\pi)^2} g_{\mu\nu} C_2. \quad (20)$$

The continuity equation of current induced in the vacuum polarization [19]

$$q^\mu \Pi_{\mu\nu}(q) = 0 \quad (21)$$

is ensured by the factor $(q_\mu q_\nu - g_{\mu\nu} q^2)$. So we set $C_2 = 0$. Consider the scattering between two electrons via the exchange of a photon with momentum transfer $q \rightarrow 0$ [19]. Adding the contribution of $\Pi_{\mu\nu}(q)$ to tree diagram amounts to modify the charge square:

$$e^2 \rightarrow e_R^2 = Z_3 e^2$$

$$Z_3 = 1 + \frac{\alpha}{3\pi} \left(\ln \frac{m^2}{\mu_3^2} - \frac{q^2}{5m^2} + \dots \right). \quad (22)$$

The choice of μ_3 will be discussed later. The next term in expansion when $q \neq 0$ contributes a modification on Coulomb potential due to vacuum polarization (Uehling potential).

3. Vertex function in QED

$$\Lambda_\mu(p', p) = (-ie)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2} \gamma_\nu \frac{i}{\not{p}' - \not{k} - m} \gamma_\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\nu. \quad (23)$$

For simplicity, we consider electron being on the mass shell: $p^2 = p'^2 = m^2$, $p' - p = q$, $p \cdot q = -\frac{q^2}{2}$. Introducing the Feynman parameter $u = x + y$ and $v = x - y$, we perform a shift in momentum integration:

$$k \rightarrow K = k - (p + \frac{q}{2})u - \frac{q}{2}v.$$

Thus

$$\Lambda_\mu = -ie^2[I_3\gamma_\mu + I_4] \quad (24)$$

$$I_3 = \int_0^1 du \int_{-u}^u dv \int \frac{d^4 K}{(2\pi)^4} \frac{K^2}{(K^2 - M^2)^3} \quad (25)$$

$$M^2 = (m^2 - \frac{q^2}{4})u^2 + \frac{q^2}{4}v^2 \quad (26)$$

$$I_4 = \int_0^1 du \int_{-u}^u dv \int \frac{d^4 K}{(2\pi)^4} \frac{A_\mu}{(K^2 - M^2)^3} \quad (27)$$

$$\begin{aligned} A_\mu = & (4 - 4u - 2u^2)m^2\gamma_\mu + 2i(u^2 - u)mq^\nu\sigma_{\mu\nu} \\ & -(2 - 2u + \frac{u^2}{2} - \frac{v^2}{2})q^2\gamma_\mu - (2 + 2u)vmq_\mu \end{aligned} \quad (28)$$

Set $K^2 = K'^2 - M^2 + M^2$, then $I_3 = I'_3 - \frac{i}{32\pi^2}$. I'_3 is only logarithmically divergent and can be treated as before to be

$$I'_3 = \frac{-i}{(4\pi)^2} \int_0^1 du \int_{-u}^u dv \ln \frac{(m^2 - \frac{q^2}{4})u^2 + \frac{q^2}{4}v^2}{\mu_1^2} \quad (29)$$

with μ_1^2 an arbitrary constant. Now $q^2 = -Q^2 < 0$ ($Q^2 > 0$)

$$I_3 = \frac{-i}{(4\pi)^2} \left\{ \ln \frac{m^2}{\mu_1^2} - \frac{5}{2} + \frac{1}{\omega} F(\omega) \right\} \quad (30)$$

$$F(\omega) = \ln \frac{1 + \omega}{1 - \omega}, \quad \omega = \frac{1}{\sqrt{\frac{4m^2}{Q^2} + 1}}.$$

On the other hand, though there is no ultra-violet divergence in I_4 , it does have infrared divergence at $u \rightarrow 0$. For handling it, we introduce a lower cutoff η in the integration with respect to u

$$\begin{aligned} I_4 = & \frac{i}{2(4\pi)^2} \left\{ [4 \ln \eta + 5] \frac{4w}{Q^2} F(w) m^2 \gamma_\mu + \frac{i4w}{Q^2} F(w) m q^\nu \sigma_{\mu\nu} \right. \\ & \left. + 4(2 \ln \eta + \frac{7}{4}) w F(w) \gamma_\mu + [\frac{1}{w} F(w) - 2] \gamma_\mu \right\}. \end{aligned} \quad (31)$$

Combining Eqs. (30) and (31) into Eq. (24), one arrives at

$$\begin{aligned}\Lambda_\mu(p', p) = & -\frac{\alpha}{4\pi} \left\{ \left[\ln \frac{m^2}{\mu_1^2} - \frac{3}{2} + \frac{1}{2\omega} F(\omega) \right] \gamma_\mu - (4 \ln \eta + 5) \frac{2\omega}{Q^2} F(\omega) m^2 \gamma_\mu \right. \\ & \left. - \frac{i2\omega}{Q^2} F(\omega) m q^\nu \sigma_{\mu\nu} - 2(2 \ln \eta + \frac{7}{4}) \omega F(\omega) \gamma_\mu \right\}.\end{aligned}\quad (32)$$

When $Q^2 \ll m^2$, we get

$$\Lambda_\mu(p', p) = \frac{\alpha}{4\pi} \left(\frac{11}{2} - \ln \frac{m^2}{\mu_1^2} + 4 \ln \eta \right) \gamma_\mu + i \frac{\alpha}{4\pi} \frac{q^\nu}{m} \sigma_{\mu\nu} - \frac{\alpha}{4\pi} \left(\frac{1}{6} + \frac{4}{3} \ln \eta \right) \frac{q^2}{m^2} \gamma_\mu.$$

It means that the interaction of the electron with the external potential is modified

$$-e\gamma_\mu \rightarrow -e[\gamma_\mu + \Lambda_\mu(p', p)]. \quad (33)$$

Besides the important term $i \frac{\alpha}{4\pi} \frac{q^\nu}{m} \sigma_{\mu\nu}$ in $\Lambda_\mu(p', p)$ which emerges as the anomalous magnetic moment of electron, the charge modification here is expressed by a renormalization factor Z_1 :

$$Z_1^{-1} = 1 + \frac{\alpha}{4\pi} \left\{ \left[2 - \ln \frac{m^2}{\mu_1^2} - \frac{1}{2w} F(w) \right] + (4 \ln \eta + 5) \frac{2wm^2}{Q^2} F(w) + (2 \ln \eta + \frac{7}{4}) 2w F(w) \right\}. \quad (34)$$

The infrared term ($\sim \ln \eta$) is ascribed to the bremsstrahlung of soft photons [20,22] and can be taken care by KLN theorem [23]. We will fix μ_1 and η below.

4. Beta function at one-loop level in QED

Adding all three FDI's at one loop level to the tree diagram, we define the renormalized charge as usual [2, 20-22]:

$$e_R = \frac{Z_2}{Z_1} Z_3^{1/2} e. \quad (35)$$

But the Ward-Takahashi Identity (WTI) implies that [20-22]

$$Z_1 = Z_2. \quad (36)$$

Therefore

$$\alpha_R \equiv \frac{e_R^2}{4\pi} = Z_3 \alpha. \quad (37)$$

Then set $p^2 = m^2$ in Z_2 and $Q^2 = 0$ in Z_1 with $\mu_1 = \mu_2$, yielding

$$\ln \eta = -\frac{5}{8}. \quad (38)$$

For any value of Q , the renormalized charge reads from Eqs. (20)—(22):

$$e_R(Q) = e \left\{ 1 + \frac{\alpha}{\pi} \int_0^1 dx [(x - x^2) \ln \frac{Q^2(x - x^2) + m^2}{\mu_3^2}] \right\} \quad (39)$$

$$e_R(Q) \sim e \left\{ 1 + \frac{\alpha}{2\pi} \left[\frac{1}{3} \ln \frac{m^2}{\mu_3^2} + \frac{1}{15} \frac{Q^2}{m^2} \right] \right\} \quad (Q^2 \ll m^2). \quad (40)$$

The observed charge is defined at $Q^2 \rightarrow 0$ (Thomson scattering) limit:

$$e_{\text{obs}} = e_R|_{Q=0} = e \quad (41)$$

which dictates that

$$\mu_3 = m. \quad (42)$$

We see that $e_R^2(Q)$ increases with Q^2 . For discussing the running of α_R with Q^2 , we define the Beta function:

$$\beta(\alpha, Q) \equiv Q \frac{\partial}{\partial Q} \alpha_R(Q) \quad (43)$$

From Eq. (39), one finds:

$$\beta(\alpha, Q) = \frac{2\alpha^2}{3\pi} - \frac{4\alpha^2 m^2}{\pi Q^2} \left\{ 1 + \frac{2m^2}{\sqrt{Q^4 + 4Q^2 m^2}} \ln \frac{\sqrt{Q^4 + 4Q^2 m^2} - Q^2}{\sqrt{Q^4 + 4Q^2 m^2} + Q^2} \right\} \quad (44)$$

$$\beta(\alpha, Q) \simeq \frac{2\alpha^2}{15\pi} \frac{Q^2}{m^2}, \quad \left(\frac{Q^2}{4m^2} \ll 1 \right) \quad (45)$$

$$\beta(\alpha, Q) \simeq \frac{2\alpha^2}{3\pi} - \frac{4\alpha^2 m^2}{\pi Q^2}, \quad \left(\frac{4m^2}{Q^2} \ll 1 \right) \quad (46)$$

which leads to the well known result $\beta(\alpha) = \frac{2\alpha^2}{3\pi}$ at one loop level at $Q^2 \rightarrow \infty$.

5. The RGE in QED with contributions from 9 kinds of fermions with masses

Usually, the GRE in QED is obtained by set $Q \rightarrow \infty$ and $\alpha \rightarrow \alpha_R(Q)$ in the right hand side of Eq. (43),

$$Q \frac{\partial}{\partial Q} \alpha_R = \frac{2\alpha_R^2}{3\pi}. \quad (47)$$

Then after integration, one yields analytically (see Eq. (1)):

$$\alpha_R(Q) = \frac{\alpha}{1 - \frac{2\alpha}{3\pi} \ln \frac{Q}{m}}. \quad (48)$$

However the renormalization is forced to be made at $Q = m$ so that

$$\alpha_R|_{Q=m} = \alpha. \quad (49)$$

We are now in a position to improve the above GRE calculation in three aspects as indicated at the beginning of this paper. For constructing a new GRE, we replace the constant α in right hand side of Eq. (44) by $\alpha_R(Q)$ and add all the contributions from charged leptons and quarks together, yielding:

$$Q \frac{d}{dQ} \alpha_R(Q) = \sum_i \epsilon_i \left\{ \frac{2\alpha_R^2(Q)}{3\pi} - \frac{4\alpha_R^2(Q)m_i^2}{\pi Q} \left[1 + \frac{2m_i^2}{\sqrt{Q^4 + 4Q^2m_i^2}} \ln \frac{\sqrt{Q^4 + 4Q^2m_i^2} - Q^2}{\sqrt{Q^4 + 4Q^2m_i^2} + Q^2} \right] \right\} \quad (50)$$

where

$$\epsilon_i = \begin{cases} 1, & i = e, \mu, \tau \\ 3 \times (\frac{2}{3})^2 = \frac{4}{3}, & i = u, c, t \\ 3 \times (-\frac{1}{3})^2 = \frac{1}{3}, & i = d, s, b. \end{cases} \quad (51)$$

Adding up contributions from particles with mass $m_e, m_\mu, m_\tau, m_c = 1.031 GeV, m_b = 4.326 GeV, m_t = 175 GeV$ we calculate the running coupling constant numerically from $\alpha_R(Q = 0) = \alpha$ till

$$\alpha_R(Q = m_Z = 91.1884 GeV) = (131.51)^{-1} \quad (52)$$

in comparison with the experimental value [24],

$$\alpha_{exp}(Q = m_Z) = (128.89)^{-1}. \quad (53)$$

The remaining discrepancy is ascribed to the contribution of light quarks (u, d, s) with average mass

$$\bar{m}_q = 92 MeV, \quad q = u, d, s. \quad (54)$$

If we adopt the following values for the mass of light quark:

$$m_u = 8 MeV, \quad m_d = 10 MeV, \quad m_s = 200 MeV \quad (55)$$

which are not far from the ratios found by Yan *et al.* [25] via the analysis of mass spectrum of mesons, then the fit will be rather good. See Fig. 1.

III. RGE of RCC in QCD

1. Self-energy of quark with mass m_i

For convenience, we use the notation and diagram in Ref. [2] at one-loop level. Then the self-energy of quark with momentum p reads

$$\Sigma_i(p) = -i(A_i + B_i \not{p}). \quad (56)$$

The similar procedure as in previous section leads to the renormalization constant for wave function:

$$Z_{2i} = (1 - B_i)^{-1} \approx 1 + B_i(p, m_i) \quad (57)$$

$$Z_{2i} = 1 + \frac{\alpha_s}{4\pi} T^a T^a \left\{ \ln \frac{m_i^2}{\mu_{2i}^2} - 3 - \frac{(m_i^2 - p^2)}{p^2} \left[1 + \frac{(m_i^2 + p^2)}{p^2} \ln \frac{(m_i^2 - p^2)}{m_i^2} \right] \right\} \quad (58)$$

where $\alpha_s = \frac{g_s^2}{4\pi}$ is the strong coupling constant, $T^a T^a = \frac{4}{3}$, and μ_{2i} is an arbitrary constant like that in Eq. (7).

2. Self-energy of gluon

The combination of contributions from the gluon loop and the Faddeev-Popov ghost field leads to

$$\Pi_{\mu\nu,ab}^g(q) = \frac{i\alpha_s}{4\pi} \delta_{ab} C_A \frac{5}{3} (g_{\mu\nu} Q^2 + q_\mu q_\nu) \ln \frac{Q^2}{\mu_3^2} \quad (59)$$

where $Q^2 = -q^2 > 0$, $C_A = 3$, and μ_3 being an another arbitrary constant (See Eq. (20)).

The third contribution is coming from quark loop with mass m_i ($i = u, d, s, c, b, t$):

$$\Pi_{\mu\nu,ab}^{q_i}(q) = \frac{i\alpha_s}{\pi} \delta_{ab} (q_\mu q_\nu - g_{\mu\nu} q^2) \int_0^1 dx (x^2 - x) \ln \frac{m_i^2 + q^2(x^2 - x)}{\mu_3^2} \quad (60)$$

(the quark notation q_i should not be confused with the momentum transfer q).

Combination of Eq. (59) with (60) induces the change of α_s :

$$\alpha_s \longrightarrow Z_3 \alpha_s$$

with

$$Z_3 = 1 + \frac{\alpha_s}{4\pi} \left[-\frac{5}{3} C_A \ln \frac{Q^2}{\mu_3^2} + \sum_{i=u}^t 4 \int_0^1 dx (x - x^2) \ln \frac{m_i^2 + Q^2(x - x^2)}{\mu_3^2} \right]. \quad (61)$$

3. Vertex functions in QCD

There are two kinds of vertex function for one species of quark with mass m_i at one-loop level in QCD, $\bar{\Gamma}_{\mu i}^{(1)}(q)$ and $\bar{\Gamma}_{\mu i}^{(2)}(q)$ (see Ref. [2]):

$$\begin{aligned} \bar{\Gamma}_{\mu i}^{(1)}(q) &= \frac{\alpha_s}{4\pi} \left(\frac{C_A}{2} - T^a T^a \right) \left\{ \left[\ln \frac{m_i^2}{\mu_1^2} - \frac{3}{2} + \frac{1}{2\omega_i} F(\omega_i) \right] \gamma_\mu \right. \\ &\quad \left. - (4 \ln \eta + 5) \frac{2\omega_i}{Q^2} F(\omega_i) m_i^2 \gamma_\mu - 2 \left(2 \ln \eta + \frac{7}{4} \right) \omega_i F(\omega_i) \gamma_\mu \right\} \end{aligned} \quad (62)$$

$$\begin{aligned} \bar{\Gamma}_{\mu i}^{(2)}(q) &= \frac{\alpha_s}{4\pi} \frac{C_A}{2} \int_0^1 du \int_{-u}^{+u} dv \left\{ -3 \gamma_\mu \left(\ln \frac{M_i^2}{\mu_1^2} + \frac{1}{2} \right) \right. \\ &\quad \left. + \gamma_\mu \frac{[2u(1-u)m_i^2 + \frac{q^2}{2}(u^2 - u - v^2)]}{2M_i^2} \right\} \end{aligned} \quad (63)$$

where μ_1 (η) is an arbitrary constant introduced for dealing with the ultraviolet (infrared) divergence (see Eqs. (23) — (34)),

$$\omega_i = \frac{1}{\sqrt{1 + 4m_i^2/Q^2}}, \quad F(\omega_i) = \ln \frac{1 + \omega_i}{1 - \omega_i} \quad (64)$$

$$M_i^2 = m_i^2(1 - u)^2 + \frac{Q^2}{4}(u^2 - v^2). \quad (65)$$

Here the new renormalization method has been used and two terms related to the anomalous magnetic moment of quarks have been omitted. The two Feynman diagrams give the correction of vertex function at one-loop level

$$-ig_s T^a \gamma_\mu \longrightarrow -ig_s T^a (\gamma_\mu + \bar{\Gamma}_{\mu i}^{(1)} + \bar{\Gamma}_{\mu i}^{(2)}) = -ig_s T^a \gamma_\mu / Z_{1i}. \quad (66)$$

Then,

$$\begin{aligned}
Z_{1i}^{-1} = & 1 + \frac{\alpha_s}{4\pi} \left(\frac{C_A}{2} - T^a T^a \right) \left\{ \ln \frac{m_i^2}{\mu_1^2} - \frac{3}{2} + \frac{1}{2\omega_i} F(\omega_i) - (4 \ln \eta + 5) \frac{2m_i^2}{Q^2} \omega_i F(\omega_i) \right. \\
& - 2 \left(2 \ln \eta + \frac{7}{4} \right) \omega_i F(\omega_i) \left. \right\} + \frac{\alpha_s}{4\pi} \frac{C_A}{2} \int_0^1 du \int_{-u}^{+u} dv \left\{ -3 \ln \frac{M_i^2}{\mu_1^2} - \frac{3}{2} \right. \\
& \left. + \frac{u(1-u)m_i^2 + \frac{Q^2}{4}(u-u^2+v^2)}{m_i^2(1-u)^2 + \frac{Q^2}{4}(u^2-v^2)} \right\}.
\end{aligned} \tag{67}$$

4. Beta function at one-loop level in QCD

Combining all of the above one-loop Feynman diagrams and considering $p = \frac{q}{2}$ in Z_{2i} , the strong coupling constant α_s is modified to

$$\alpha_s \longrightarrow \alpha_{si}(Q, m_i) = \frac{Z_{2i}^2 Z_3}{Z_{1i}^2} \alpha_s. \tag{68}$$

For discussing the running of $\alpha_{si}(Q, m_i)$ with Q^2 , we define the β -function

$$\begin{aligned}
\beta_i(Q, m_i) &= Q \frac{\partial}{\partial Q} \alpha_{si}(Q, m_i) = 2Q^2 \frac{\partial}{\partial Q^2} \alpha_{si}(Q, m_i) \\
&= 2Q^2 \alpha_s \left(\frac{\partial}{\partial Q^2} Z_{2i}^2 + \frac{\partial}{\partial Q^2} Z_3 + \frac{\partial}{\partial Q^2} Z_{1i}^{-2} \right).
\end{aligned} \tag{69}$$

By denoting

$$\begin{aligned}
\frac{\partial}{\partial Q^2} Z_{2i}^2 &= \frac{\alpha_s}{4\pi Q^2} B_{2i}(Q, m_i) \\
\frac{\partial}{\partial Q^2} Z_3 &= \frac{\alpha_s}{4\pi Q^2} B_3(Q, m_u, \dots, m_t) \\
\frac{\partial}{\partial Q^2} Z_{1i}^{-2} &= \frac{\alpha_s}{4\pi Q^2} B_{1i}(Q, m_i),
\end{aligned} \tag{70}$$

we get

$$\beta_i(Q, m_i) = \frac{\alpha_s^2}{2\pi} (B_{1i} + B_{2i} + B_3). \tag{71}$$

5. RGE for quark q_i with mass m_i in QCD

The RGE is established by simply substituting the α_s by $\alpha_{si}(Q, m_i)$ at the right side, yielding

$$Q \frac{\partial}{\partial Q} \alpha_{si}(Q, m_i) = \frac{1}{2\pi} (B_{1i} + B_{2i} + B_3) \alpha_{si}^2(Q, m_i). \tag{72}$$

IV. Numerical calculation of RGE in QCD

Obviously, Eq. (72) can only be integrated numerically for one species of quark with mass m_i . We adopt the experimental data $Q = m_Z = 91.1884 \text{ GeV}$, $\alpha_{si} = 0.118$ [26,27] as the initial value of integration. Then, $\alpha_{si}(Q, m_i)$ becomes

$$\alpha_{si}(Q, m_i) = \frac{1}{\frac{1}{0.118} + \frac{1}{2\pi} \int_Q^{91188.4} (B_{1i} + B_{2i} + B_3) \frac{1}{Q} dQ} \quad (73)$$

where

$$B_{1i}(Q, m_i) = \frac{1}{3} \left(\frac{m_i^2}{Q^2} \omega_i F(\omega_i) + \frac{m_i^2}{Q^2} \left(1 - \frac{4m_i^2}{Q^2} \right) \omega_i^3 F(\omega_i) + \left(\frac{1}{2} - \frac{2m_i^2}{Q^2} \right) \omega_i^2 + \frac{1}{2} \right) - 9 + 3 \int_0^1 du G_i(u, Q) \quad (74)$$

$$G_i(u, Q) = \frac{4m_i^2}{Q^2} (1-u)(u - 2u^2 - \frac{1}{2\xi_i}) \ln \frac{\xi_i + u}{\xi_i - u} + \frac{1}{\xi_i^2} \left(u^2 + \frac{4m_i^2}{Q^2} u(1-u) \right) \quad (75)$$

$$\xi_i = \sqrt{\frac{4m_i^2}{Q^2} (1-u)^2 + u^2}, \quad \omega_i = \frac{1}{\sqrt{1 + 4m_i^2/Q^2}}, \quad F(\omega_i) = \ln \frac{1 + \omega_i}{1 - \omega_i}, \quad (76)$$

$$B_{2i}(Q, m_i) = \frac{8}{3} \left(1 + \frac{8m_i^2}{Q^2} \left(-1 + \frac{4m_i^2}{Q^2} \ln \left(1 + \frac{Q^2}{m_i^2} \right) \right) \right) \quad (77)$$

$$B_3(Q, m_u, \dots, m_t) = -1 - \sum_{i=u}^t \left(\frac{4m_i^2}{Q^2} - \frac{8m_i^4}{Q^4} \omega_i F(\omega_i) \right). \quad (78)$$

The results are shown in Figures 2 and 3.

V. Summary and discussion

1. Let us first check the zero mass limit of above equations for returning to the familiar result Eq. (2). For the purpose we look directly at the Z_i in the limit $m_i/Q \rightarrow 0$, yielding

$$\begin{cases} Z_1^{-1} = 1 - \frac{\alpha}{4\pi} (C_A + T^a T^a) \ln \frac{Q^2}{\mu^2} \\ Z_2 = 1 + \frac{\alpha}{4\pi} T^a T^a \ln \frac{Q^2}{\mu^2} \\ Z_3^{1/2} = 1 + \frac{\alpha}{8\pi} \left(\frac{4}{3} C_f - \frac{5}{3} C_A \right) \ln \frac{Q^2}{\mu^2} \end{cases} \quad (79)$$

where we have chosen $\ln \eta = -1$ with another constants $\mu_1 = \mu_2 = \mu_3 = \mu$. This recipe amounts to define the value of α_s at high Q limits.

Substituting Eq. (79) into Eq. (71), we obtain

$$\beta(Q) = -\frac{\alpha^2}{2\pi}\beta_0, \quad \beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f, \quad C_A = 3. \quad (80)$$

Then the RGE reads

$$Q \frac{\partial}{\partial Q} \alpha_R(Q) = -\frac{1}{2\pi} \beta_0 \alpha_R^2(Q) \quad (81)$$

with its solution precisely giving Eq. (2).

2. Alternatively, we manage to keep the quark mass in all B_i to get the RGE (72) before setting the limit $m_i \rightarrow 0$:

$$\begin{aligned} B_2 &\rightarrow 2T^a T^a \\ B_3 &\rightarrow -\frac{5}{3}C_A + \frac{2}{3}n_f \\ B_1 &\rightarrow 2\left(\frac{C_A}{2} - T^a T^a\right) - 2C_A. \end{aligned}$$

Thus, in the limit $m_i \rightarrow 0$,

$$B_1 + B_2 + B_3 \rightarrow \frac{2}{3}n_f - \frac{8}{3}C_A = -\beta'_0. \quad (82)$$

It is interesting to compare (82) with (80), showing that

$$\beta_0 - \beta'_0 = C_A \quad (83)$$

which is stemming from the different order of taking limit: either $m_i \rightarrow 0$ before the derivative $\frac{\partial}{\partial Q^2}$ or vice versa.

3. But the zero mass limit is certainly not a good one as discussed in the introduction. And this is why one usually had to take $n_f = 3$ in β_0 . The mass of c or b quark is too heavy to be neglected. Therefore, we have calculated seriously the RGE for five quarks (u, d, s, c, b) with masses except t quark. The latter is too heavy to be created explicitly in the energy region considered. Notice that, however, the contribution of t quark is still existing in the function B_3 , Eq. (78).

4. The prominent feature of our RGE calculation is the following:

(a) The RCC $\alpha_{si}(Q, m_i)$ has a flavor dependence, i.e., it is different for different quark with different m_i .

(b) The value of $\alpha_{si}(Q, m_i)$ increases from normalized value 0.118 at $Q = M_Z = 91.1884 GeV$ with the decrease of Q until a maximum α_{si}^{max} is reached at $Q = \Lambda_i$. The smaller the m_i is, the smaller the Λ_i is and the higher the value of α_{si}^{max} will be. When $Q \rightarrow 0$, all α_{si} approach to zero.

(c) The value of Λ_i could be explained as the existence of a critical length scale L_i of $q_i\bar{q}_i$ pair

$$L_i \sim \hbar/\Lambda_i \quad (84)$$

while the value α_{si}^{max} may correspond to the excitation energy for breaking the binding $q_i\bar{q}_i$ pair, i.e., the threshold energy scale against its dissociation into two bosons:

$$E_i^{thr} \sim \alpha_{si}^{max}/L_i \sim \alpha_{si}^{max}\Lambda_i/\hbar. \quad (85)$$

The numerical estimation of these values is listed at the table 1. It is interesting to see that E_i^{thr} for u, d quarks is of the order of π meson while that for c or b quark could be compared with the D^+D^- or B^+B^- threshold respectively.

	u	d	s	c	b
$m_i c^2 (\text{MeV})$	8	10	200	1031	4326
$\Lambda_i (\text{MeV})$	18.4	18.4	290	1640	7040
α_{si}^{max}	12.43	9.368	0.3027	0.2038	0.1610
$L_i (\text{fm})$	10.73	10.73	0.6809	0.1204	0.02805
$E_i^{thr} (\text{MeV})$	228.7	172.4	87.77	334.3	1133

Table 1

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Figure Caption

Figure 1:

The nine curves (see from the lowest) represent respectively the contributions to the running electromagnetic coupling constant from

- (1) electron e only,
- (2) e and muon (μ) only,
- (3) all charged leptons e , μ and τ only,
- (4) e , μ , τ and c quark only,

- (5) e, μ, τ, c and b quark only,
- (6) e, μ, τ, c, b and t quark only,
- (7) e, μ, τ, c, b, t and u quark only,
- (8) e, μ, τ, c, b, t, u and d quark only,
- (9) all charged leptons and quarks.

The last curve is actually coinciding with the experimental curve denoted by dot line which can also be fitted by assuming three light quarks (u, d, s) having average mass $92\text{MeV}/c^2$.

Figure 2:

The running strong coupling constant curves for u and d quarks.

Figure 3:

The running strong coupling constant curves for s, c and b quarks.

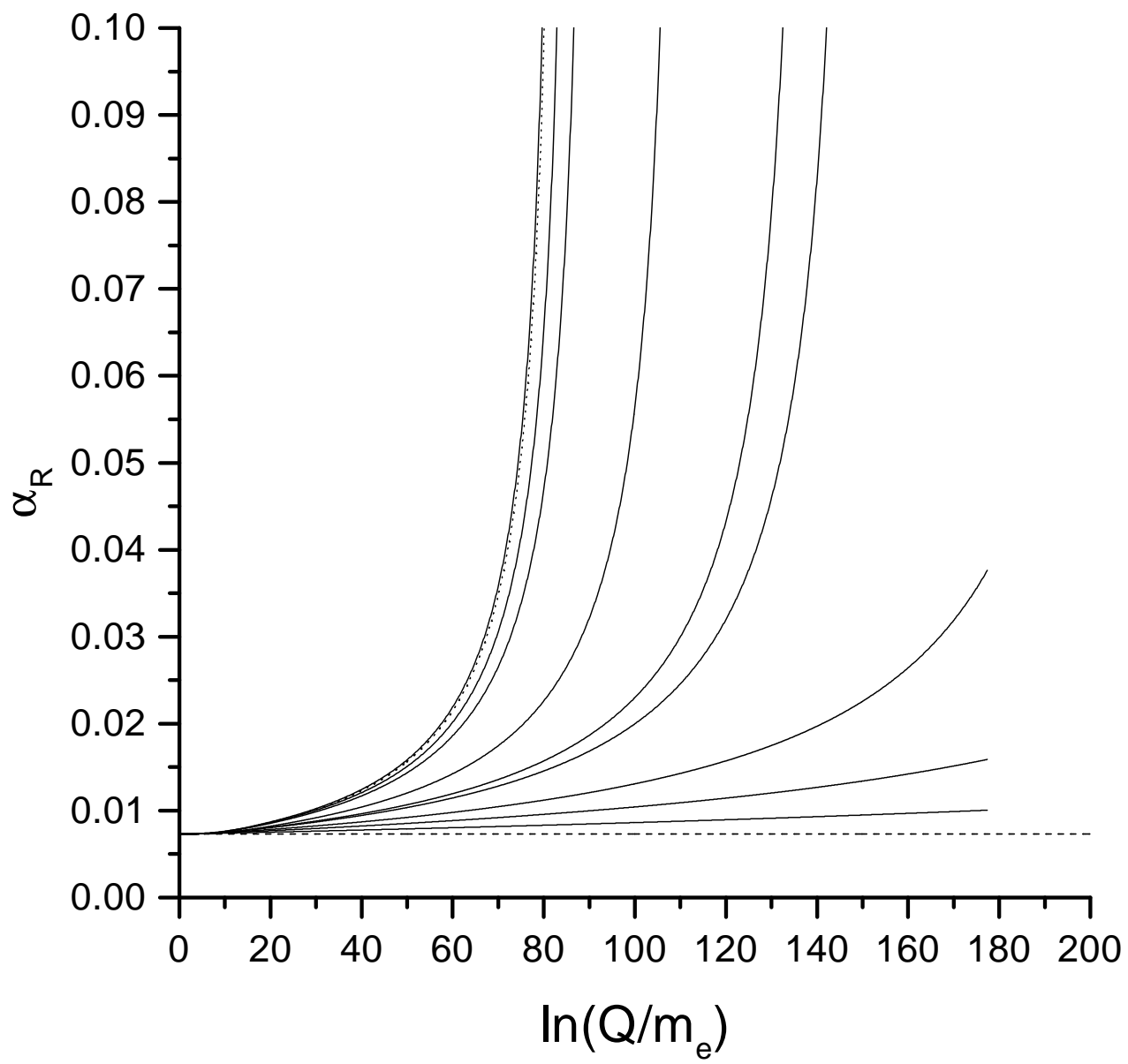


Figure 1

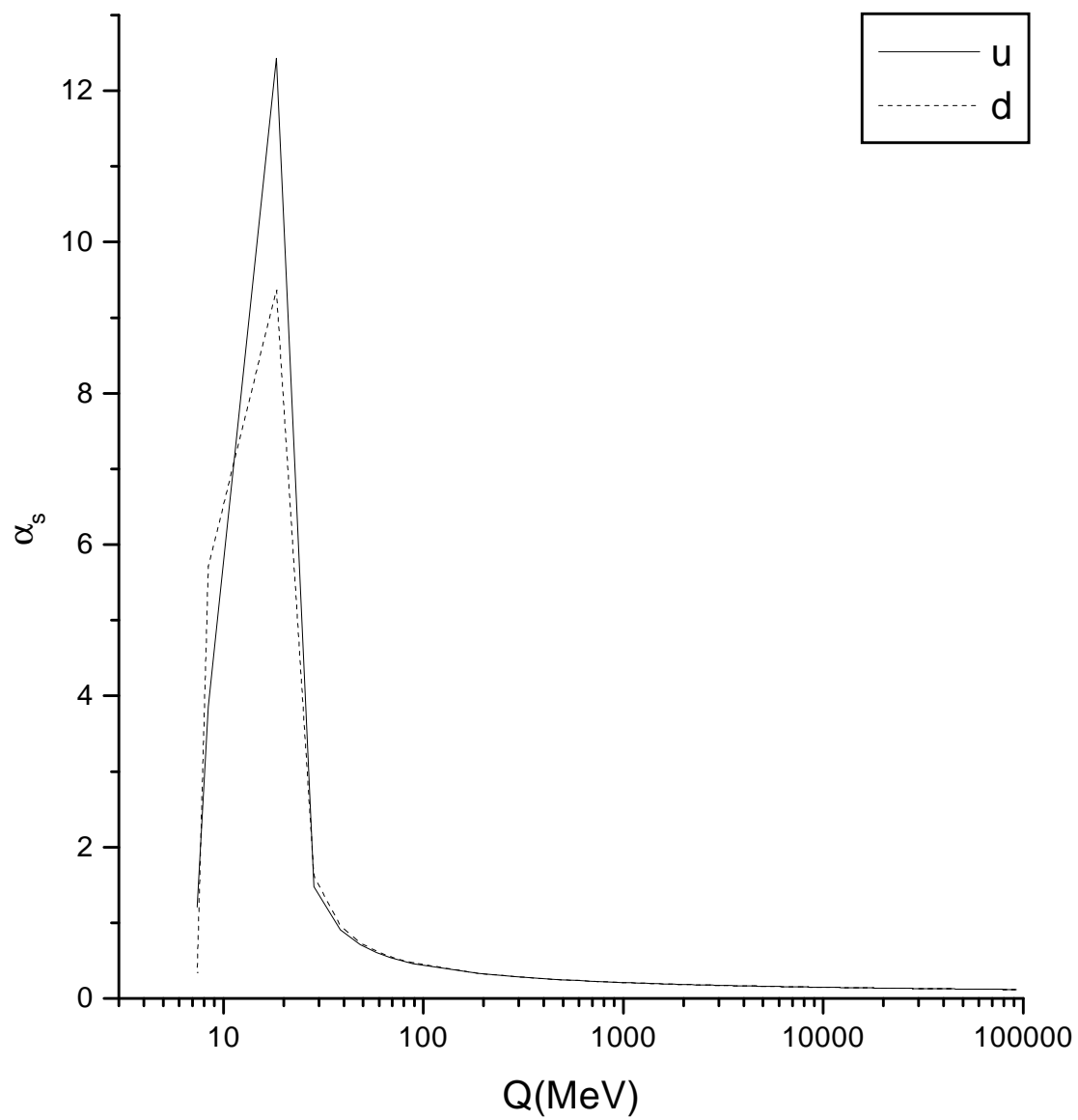


Figure 2

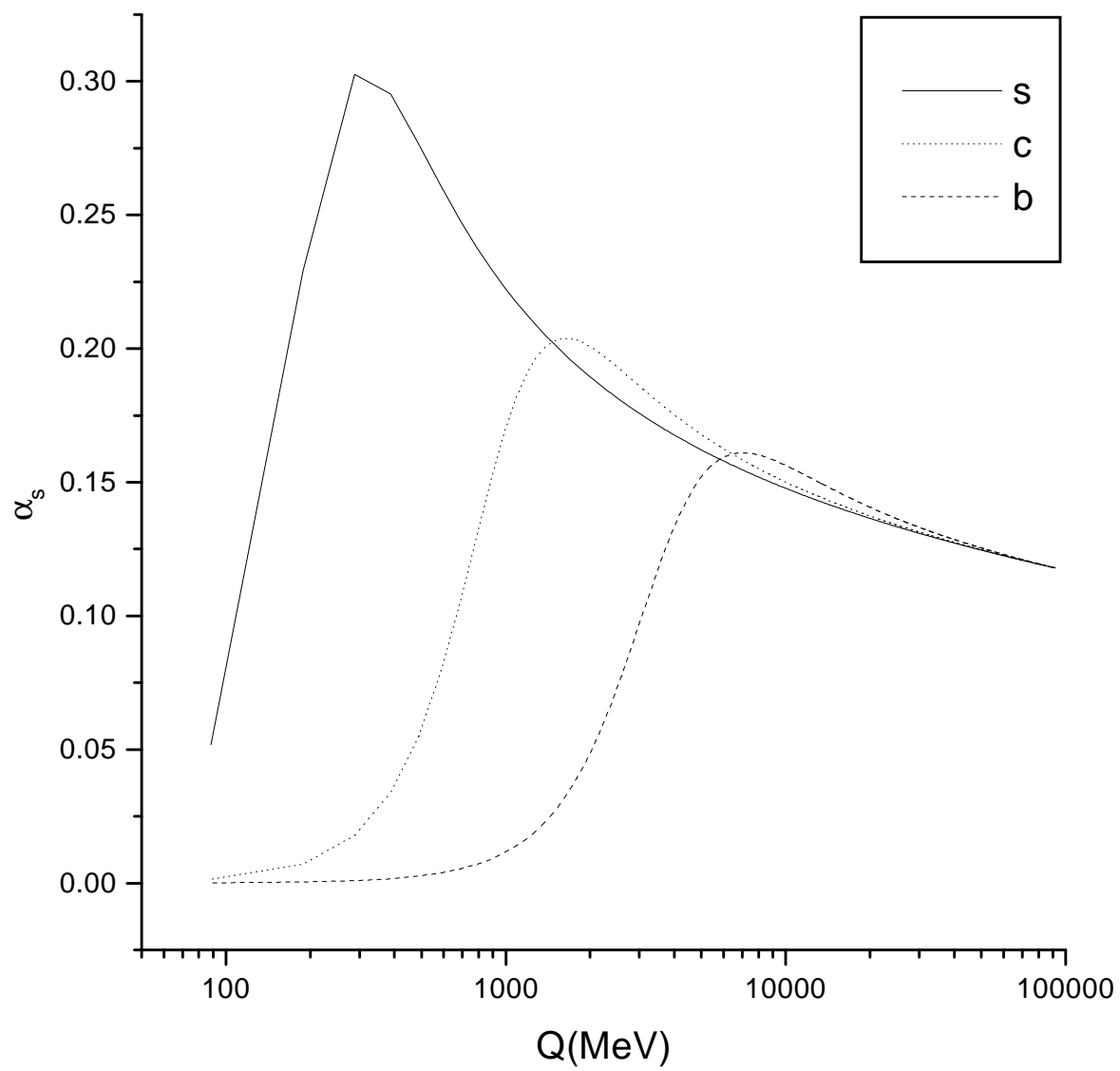


Figure 3